A NEW APPROACH TO INFERENCE FROM ACCELERATED LIFE TESTS

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1. Introduction.

In practice, long life items are subjected to larger than normal stresses in order to obtain failure data in a short amount of test time. Such tests are called accelerated or overstress life tests, and the goal is to make inferences about the life distribution of the items at the normal stress levels using failure data from accelerated tests.

The current approach to this problem involves making assumptions about the distribution of failure times, and about the functional relationship between the parameters of the failure distribution and the applied stress. Such a relationship is known as an acceleration function or a time transformation function; examples of these are, the Power Law, the Arrhenius Law, the Eyring Law, etc.. Another assumption commonly made states that at all stress levels, the failure times are governed by distributions which are members of the same parametric family, such as an exponential or a Weibull. These assumptions, though appealing from a statistical point of view, may be unreasonable in many practical situations. Of particular concern is the last assumption. The different stress levels may have different effects on the "failure mechanisms", and thus from an engineering point of view, it may be more realistic to allow for different forms of the failure distribution at the different stress levels.

In a recent paper, Proschan and Singpurwalla (1978) (hereafter referred to as PS(1978)) have proposed a new approach for making inferences from accelerated life tests which requires neither distributional assumptions nor the specification of a time transformation function. Rather, their

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approach is Bayesian, and is motivated by procedures actually used in practice by reliability engineers. The Bayesian point of view enables a user to incorporate some a priori information which is available in accelerated life tests.

In Section 2 we shall review this pragmatic Bayesian approach to accelerated life testing by PS(1978), and present it in a manner which makes it easy to understand. In Section 3 we shall demonstrate the usefulness of this approach by applying it to some real life data on accelerated life tests presented in Nelson (1970).

Our objective in writing this paper is to make this new approach to accelerated life testing accessible to the reliability engineer, and to demonstrate its usefulness by applying it to a realistic situation.

2. **A Pragmatic Bayesian Approach to Accelerated Life Testing.**

We shall denote the k accelerated stresses (environments) by $E_1, E_2, \ldots, E_k$, and the normal or use conditions stress (environment) by E_u. Let

$$E_1 > E_2 > \cdots > E_k > E_u , \qquad (2.1)$$

where, $E_i > E_j$ denotes the fact that E_i is more severe than E_j.

Let F_j be the failure distribution of the items tested under environment E_j, and let $\lambda_j(x)$ denote the failure rate of F_j at time point $x \geq 0$.

Because of condition (2.1) it is logical from a physical point of view to assume that for any $x \geq 0$,

$$\lambda_1(x) \geq \lambda_2(x) \geq \cdots \geq \lambda_k(x) \geq \lambda_u(x) \ . \tag{2.2}$$

Using failure data obtained under $E_1, E_2, \ldots, E_k$, we would like to obtain $\hat{\lambda}_j(x)$, an estimate of $\lambda_j(x)$, $j = 1, 2, \ldots, k$, such that for some $0 \leq L < \infty$, and all $x \in [0, L]$,

$$\hat{\lambda}_1(x) \overset{st}{\geq} \hat{\lambda}_2(x) \overset{st}{\geq} \cdots \overset{st}{\geq} \hat{\lambda}_k(x) \ . \tag{2.3}$$

The notation $X \overset{st}{\geq} Y$ denotes the fact that X is <u>stochastically larger</u> than Y; that is

$$P[X \geq x] \geq P[Y \geq x] \quad \text{for all } x \ .$$

In order to obtain estimators $\hat{\lambda}_j(x)$, $j = 1, 2, \ldots, k$, which satisfy (2.3), we shall use a Bayesian approach. Under this approach, condition (2.2) is incorporated as a prior assumption. However, we shall first define the "average failure rate" and discuss its Bayesian estimation unconstrained by (2.2).

2.1. A Bayesian Estimation of the Average Failure Rate.

Let $N_j(t)$ be the number of items undergoing life test at time t, under environment E_j. Divide the time interval $[0, L]$ into sub-intervals of length $h > 0$, where h is chosen to make L a multiple of h. Thus, the total number of sub-intervals is L/h.

For convenience, we shall introduce the following notation:

$\quad t_i$ = the time point $(i-1)h$, $i = 1, 2, \ldots, (L/h)$

$\quad [t_i, t_i + h)$ = the ith time interval

$N_{j,i}$ = the number of items exposed to the environment E_j at time t_i

$\lambda_{j,i}$ = the failure rate of F_j at t_i

$x_{j,i}$ = the number of failures in the ith time interval under environment E_j

$p_{j,i}$ = the probability of failure of a unit in the ith time interval under environment E_j.

Let

$$p^*_{j,i} = \frac{p_{j,i}}{1 - \sum_{\ell=1}^{i-1} p_{j,\ell}} \; ;$$

then $p^*_{j,i}$ is the conditional probability that an item which has survived to time t_i will fail by time $t_i + h$. We can also interpret $p^*_{j,i}$ as the <u>average</u> <u>failure</u> <u>rate</u> over the interval $[t_i, t_i + h)$.

If we assume that there are no withdrawals, removals, or censoring, then

$$N_{j,i+1} = N_{j,i} - x_{j,i} \; , \qquad i = 1,2,\ldots,(L/h) \; .$$

Our Bayesian analysis involves assigning prior distributions to the average failure rates. Suppose that the $p^*_{j,i}$, $i = 1,2,\ldots,(L/h)$, are a priori independent beta random variables with prior parameters α and β_j, so that their marginal densities are

$$f(p^*_{j,i}) = \frac{\Gamma(\alpha+\beta_j)}{\Gamma(\alpha)\Gamma(\beta_j)} (p^*_{j,i})^{\alpha-1} (1-p^*_{i,j})^{\beta_j-1} \; .$$

Then, given the $N_{j,i}$'s and the $x_{j,i}$'s, the posterior density of $(p^*_{j,1}, p^*_{j,2}, \ldots, p^*_{j(L/h)})$ is [cf. PS(1978)]

$$\prod_{i=1}^{(L/h)} \frac{\Gamma(\alpha+\beta_j+N_{j,i})}{\Gamma(\alpha+x_{j,i})\Gamma(\beta_j+N_{j,i}-x_{j,i})} \, (p^*_{j,i})^{\alpha+x_{j,i}-1} (1-p^*_{j,i})^{\beta_j+N_{j,i}-x_{j,i}-1} \, . \quad (2.4)$$

2.2. A Bayesian Estimation of the Ordered Average Failure Rates.

In the Bayesian context, condition (2.2) leads us to the requirement that for every fixed value of i,

$$p^*_{j-1,i} \overset{st}{\geq} p^*_{j,i} \quad \text{for} \quad j = 2,3,\ldots,k \, .$$

Thus, the prior distributions of $p^*_{j-1,i}$ and $p^*_{j,i}$ will have to be chosen so as to ensure that

$$P[p^*_{j-1,i} \geq p] \geq P[p^*_{j,i} \geq p] \quad \text{for all} \quad p, \ 0 \leq p \leq 1 \, . \quad (2.5)$$

One way of achieving the above condition is to require [see PS(1978), Appendix A] that

$$\beta_j \geq \beta_{j-1} \quad \text{for} \quad j = 2,3,\ldots,k \, .$$

To be assured that condition (2.5) is also satisfied with respect to the posterior distributions of $p^*_{j-1,i}$ and $p^*_{j,i}$, it is _sufficient_ that for every fixed value of i,

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$$x_{j,i} \leq x_{j-1,i}$$

and

$$\beta_j + N_{j,i} - x_{j,i} \geq \beta_{j-1} + N_{j-1,i} - x_{j-1,i}$$

for $j = 2,3,\ldots,k$.

The first part of condition (2.6) states that the number of failures in the interval $[t_i, t_i + h)$ under environment E_j must not be greater than the number of failures in the same interval under environemnt E_{j-i}, for all values of j. Furthermore, as is discussed in PS(1978) a reasonable strategy is to have more items initially on test under the more severe environment E_{j-1} than under the environment E_j, so that $N_{j,1} < N_{j-1,1}$ for all j.

Since $N_{j,i} = N_{j,i-1} - x_{j,i-1}$, the second part of condition (2.6) can be written as

$$\beta_{j-1} + N_{j-1,i+1} \leq \beta_j + N_{j,i+1} \quad \text{for} \quad j = 2,3,\ldots,k .$$

Thus for every fixed value of i, the number surviving at the start of the $(i+1)^{st}$ interval plus the prior parameter β_{j-1} for the environment E_{j-1} must be not greater than the corresponding sum for the environment E_j. Since the number of failures in a particular interval is a function of the severity of the environment and the number on test, and since $\beta_{j-1} \leq \beta_j$, a reasonable strategy is to choose $\beta_{j-1} < \beta_j$.

Thus, the prior parameters β_{j-1} and β_j are indicative of the relative severity of the environmental conditions E_{j-1} and E_j. Because of condition (2.1):

$$\beta_1 < \beta_2 < \cdots < \beta_k ,$$

with the values of the β_j's being indicative of the severity of the E_j's.

If the prior parameters β_j, the failure data $x_{j,i}$, and the $N_{j,i}$ are such that condition (2.6) is satisfied for every fixed value of i, then the stochastic ordering condition (2.5) will be automatically satisfied with respect to the posterior distribution of $p^*_{j,1}, \ldots, p^*_{j,(L/h)}$. If this is not the case, then we shall "pool" the adjacent violators using the pooling procedure described below; the purpose is to eliminate violations of (2.6).

2.3. The Pooling of Adjacent Violators.

The procedure for pooling adjacent violators described here is commonly used in isotonic regression; see Barlow, Bartholomew, Bremner and Brunk (1972).

Consider the time interval $[(i-1)h, ih)$; by condition (2.6) we require

$$x_{1,i} \geq x_{2,i} \geq \cdots \geq x_{j-1,i} \geq x_{j,i} \geq \cdots \geq x_{k,i}$$

and

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$$\beta_1 + N_{1,i} - x_{1,i} \le \beta_2 + N_{2,i} - x_{2,i} \le \cdots \le \beta_{j-1} + N_{j-1,i} - x_{j-1,i}$$

$$\le \beta_j + N_{j,i} - x_{j,i} \le \cdots \le \beta_{jk} + N_{k,i} - x_{ki} \; .$$

If a reversal occurs, that is, if either

$$x_{j-1,i} < x_{j,i}$$

or if

$$\beta_{j-1} + N_{j-1,i} - x_{j-1,i} > \beta_j + N_{j,i} - x_{j,i} \; ,$$

then we pool the violators and replace them as shown below.

Replace both $x_{j-1,i}$ and $x_{j,i}$ by $\frac{1}{2}(x_{j-1,i} + x_{j,i})$ and $\beta_{j-1} + N_{j-1,i} - x_{j-1,i}$ and $\beta_j + N_{j,i} - x_{j,i}$ by $\frac{1}{2}(\beta_{j-1} + \beta_j + N_{j-1,i} + N_{j,i} - x_{j-1,i} - x_{j,i})$.

We now test the new sequences to see if they are properly ordered. If not, we replace the adjacent violators by their appropriate averages. Thus, if

$$\frac{1}{2}(x_{j-1,i} + x_{j,i}) = \frac{1}{2}(x_{j-1,i} + x_{j,i}) < x_{j+1,i} \; ,$$

then we replace each of the three by the average

$$\frac{1}{3}(x_{j-1,i} + x_{j,i} + x_{j+1,i}) \; .$$

The same procedure is used for the $(\beta_j + N_{j,i} - x_{j,i})$'s.

We follow the above procedure for all time intervals and continue until all reversals are eliminated. Note that excessive pooling will occur if the relationship (2.2) does not actually hold, or if the environmental conditions are too similar to each other.

Assuming a squared error loss, before pooling, the Bayes estimator of $p^*_{j,i}$ is [see PS(1978)]

$$\hat{p}^*_{j,i} = \frac{\alpha + x_{j,i}}{\alpha + \beta_j + N_{j,i}} . \tag{2.7}$$

If we have done some pooling, then the $x_{j,i}$'s and the $(\beta_j + N_{j,i})$'s are replaced by their appropriate pooled averages.

2.4. A Model for Extrapolation to Use Condition Stress.

We shall use the $\hat{p}^*_{j,i}$'s to estimate $p^*_{u,1}, p^*_{u,2}, \ldots, p^*_{u,(L/h)}$, the average failure rates under the use conditions environment E_u.

In the absence of any physical or engineering knowledge about the relationship between the average failure rate and the corresponding stress, we shall postulate the following simple but reasonable relationship among the average failure rates.

For each value of i, $i = 1, 2, \ldots, (L/h)$,

$$p^*_{k,i} = w_0 + w_1 p^*_{k-1,i} + \cdots + w_{k-1} p^*_{1,i} , \tag{2.8}$$

where $w_0, w_1, \ldots, w_{k-1}$ are unknown constants.

The above relationship states that the average failure rate over a particular time interval under the environment E_k is a weighted sum of the average failure rates over the same time interval under the environments $E_{k-1}, E_{k-2}, \ldots, E_1$.

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Let $\hat{w}_0, \hat{w}_1, \ldots, \hat{w}_{k-1}$, be the least-squares estimators of $w_0, w_1, \ldots, w_k$; these can be obtained routinely from equation (2.8), but with $\hat{p}^*$ in place of p^* throughout. Thus for $i = 1, 2, \ldots, (L/h)$, we have

$$\hat{p}^*_{u,i} = \hat{w}_0 + \hat{w}_1 \hat{p}^*_{k,i} + \cdots + \hat{w}_{k-1} \hat{p}^*_{2,i} \qquad (2.9)$$

as the estimators of the average failure rate under E_u. As a consequence, we also have

$$\hat{p}^*_{u,(L/h)+1} = 1 - \sum_{i=1}^{(L/h)} \hat{p}^*_{u,i} \ .$$

An estimator of $\overline{F}_u(t)$, the probability of an item surviving to time t under E_u, the use conditions stress, is

$$\hat{\overline{F}}_u(t) = \prod_{i=1}^{t/h} (1 - \hat{p}^*_{u,i}) \ ,$$

where the $\hat{p}^*_{u,i}$ are given by (2.9).

3. An Illustrative Real Life Example.

In this section we shall apply the methodology discussed in the previous section to some accelerated life test data given by Nelson (1970). These data represent the times to breakdown of an insulating fluid subjected to elevated voltage stress levels. For convenience, we shall consider here only 4 accelerated voltage levels - 36, 34, 32, and 30 kilovolts (KV's). The failure times (in minutes) under the various

stress levels are given in Table 1. Nelson's original data correspond to 7 different stress levels, but some of these contain very few failure times and are therefore omitted.

We shall assume that the use conditions stress is 28 KV, and apply our approach to estimate the failure distribution of the breakdown times at this stress.

Following the notation of Section 2, we shall choose L to be 100 minutes, and h to be 1 minute, making the total number of time intervals equal to 100. For our prior parameters we shall, following the discussion in Sections 2.1 and 2.2, choose $\alpha = 1$, with $\beta_1 = 1$, $\beta_2 = 2$, $\beta_3 = 7$, and $\beta_4 = 12$. The above choice of our prior parameters β_1, β_2, β_3, and β_4 is motivated by an inspection of the failure times in Table 1. Observe that even though the stress levels decrease successively by only 2 KV's, the corresponding change in the failure times from one stress level to the next lower appears to be more drastic. This may be due to a change in the failure mechanism as we go to the lower stresses. Note that $N_1 = 15$, $N_2 = 19$, $N_3 = 15$, and $N_4 = 11$.

After computing the $x_{j,i}$'s and the $(\beta_j + N_{j,i} - x_{j,i})$'s for $j = 1,2,3,4$, $i = 1,2,\ldots,100$, and after pooling the adjacent violators as discussed in Section 2.3, we obtain the Bayes estimators $\hat{p}^*_{j,i}$ using Equation (2.7). These values are given in Table 2, wherein, for convenience, we have limited ourselves to time intervals of width 5. Observe that the pooling scheme ensures that for each time interval the $\hat{p}^*_{j,i}$'s decrease with increasing stress level. For example, $\hat{p}^*_{1,20} = .3333$, $\hat{p}^*_{2,20} = .1250$, $\hat{p}^*_{3,20} = .0714$, and $\hat{p}^*_{4,20} = .0454$.

Our next step is to use the entries in Table 2 to obtain estimates of the w's given in Equation (2.8). If we designate the 30 KV level as the level $k(=4)$ of Equation (2.8), and the 32, 34, and the 36 KV levels as k-1, k-2, and k-3, respectively, then, the least squares estimation of the w's gives us for $i = 1,2,\ldots,100$,

$$\hat{p}^*_{4,i} = .2360 + .18861\hat{p}^*_{3,i} + .13538\hat{p}^*_{2,i} + .09857\hat{p}^*_{1,i} .$$

We shall now use the following equation to estimate $\hat{p}^*_{u,i}$, $i = 1,2,\ldots,100$, where $u(=5)$ denotes the use condition stress level of 28 KV:

$$\hat{p}^*_{5,i} = .2360 + .1886\hat{p}^*_{4,i} + .13538\hat{p}^*_{3,i} + .09857\hat{p}^*_{2,i} .$$

An estimate of the survival probability at use conditions voltage of 28 KV is given by

$$\hat{\bar{F}}_u(t) = \prod_{i=1}^{t} (1-\hat{p}^*_{5,i}) .$$

The values of $\hat{\bar{F}}_u(t)$, $t = 5,10,\ldots,100$, are given in Table 3, and plotted in Figure 1.

Table 1. <u>Times to Breakdown of an Insulating Fluid (in Minutes) Under Various Values of the Stress (in Kilovolts).</u>
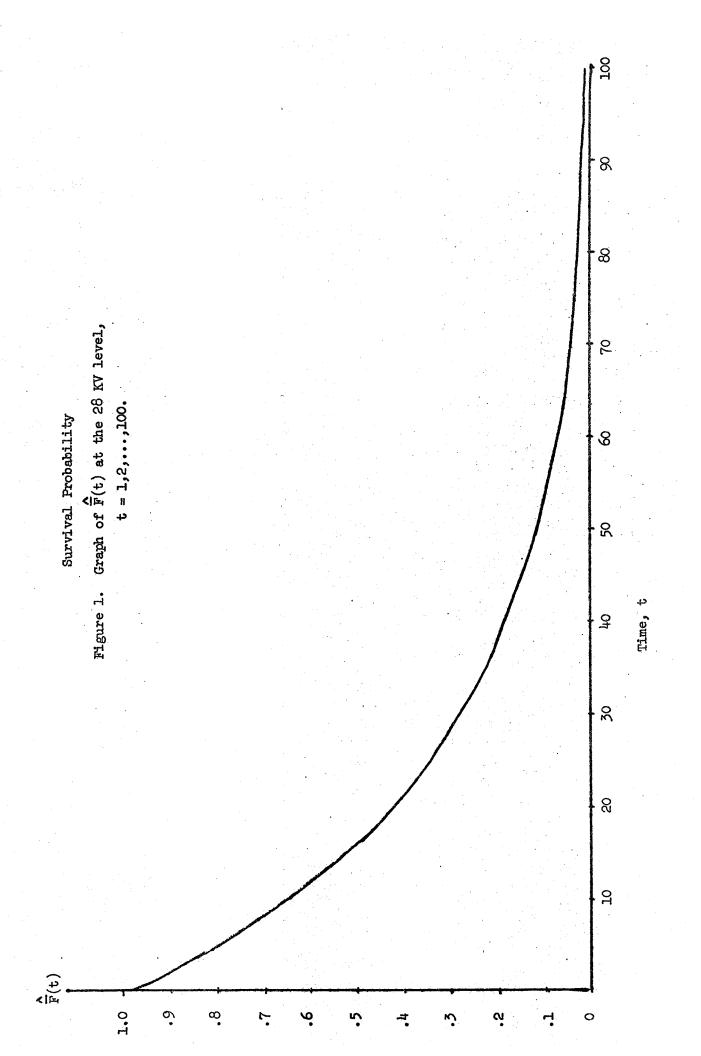
<u>36KV</u>	<u>34KV</u>	<u>32KV</u>	<u>30KV</u>
.35	.19	.27	7.74
.59	.78	.40	17.05
.96	.96	.69	20.46
.99	1.31	.79	21.02
1.69	2.78	2.75	22.66
1.97	3.16	3.91	43.40
2.07	4.15	9.88	47.30
2.58	4.67	13.95	139.07
2.71	4.85	15.93	144.12
2.90	6.50	27.80	175.88
3.67	7.35	53.24	194.90
3.99	8.01	82.85	
5.35	8.27	89.29	
13.77	12.06	100.58	
25.50	31.75	215.10	
	32.52		
	33.91		
	36.71		
	72.89		

Table 2. Estimated Values $\hat{p}^*_{j,i}$ of the Average Failure Rates Under Various Values of the Stress (in Kilovolts).

| | Stress | | | |
Time Interval	36KV	34KV	32KV	30KV
5	.3846	.1724	.0588	.0417
10	.3077	.1429	.0816	.0435
15	.3333	.1250	.0667	.0435
20	.3333	.1250	.0714	.0454
25	.3333	.1250	.0714	.0526
30	.5000	.1250	.0769	.0526
35	.5000	.2000	.0769	.0526
40	.5000	.2500	.0769	.0526
45	.5000	.2500	.0769	.0556
50	.5000	.2500	.0769	.0588
55	.5000	.2500	.0833	.0588
60	.5000	.2500	.0833	.0588
65	.5000	.2500	.0833	.0588
70	.5000	.2500	.0833	.0588
75	.5000	.3333	.0833	.0588
80	.5000	.3333	.0833	.0588
85	.5000	.3333	.0909	.0588
90	.5714	.4000	.1290	.0588
95	.5000	.3333	.1000	.0588
100	.5000	.3333	.1000	.0588

Table 3. Values of $\hat{\bar{F}}(t)$ at the 28 KV level.

t	$\hat{\bar{F}}(t)$
5	.803
10	.650
15	.525
20	.424
25	.343
30	.277
35	.222
40	.177
45	.141
50	.112
55	.090
60	.071
65	.057
70	.045
75	.036
80	.028
85	.022
90	.017
95	.014
100	.011

Survival Probability

Figure 1. Graph of $\hat{\bar{F}}(t)$ at the 28 KV level,
$t = 1,2,\ldots,100.$

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Abstract

In a recent paper, Proschan and Singpurwalla (1978) have proposed a new approach for making inferences from accelerated life tests. Their approach is significantly different from those that have been considered in the past, and is motivated by what is actually done in practice. A priori information which is generally available to the engineer is incorporated under their approach by adopting a Bayesian point of view. The usual assumptions about the failure distributions and the acceleration functions, which are appealing from a statistical point of view, have been sacrificed to achieve greater reality.

In this paper, we apply this new approach to some real life data arising from an accelerated life test. In the process, we explain our new approach in a manner which makes it easy to understand and apply by the reliability practitioner.